Tensors from K3 Orientifolds

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Abstract

Recently Gimon and Johnson [1] and Dabholkar and Park [2] have constructed Type I theories on K3 orbifolds. The spectra differ from that of Type I on a smooth K3, having extra tensors. We show that the orbifold theories cannot be blown up to smooth K3's, but rather $\mathbf{Z}_2$ orbifold singularities always remain. Douglas's recent proposal to use D-branes as probes is useful in understanding the geometry. The $\mathbf{Z}_2$ singularities are of a new type, with a different orientifold projection from those previously considered. We also find a new world-sheet consistency condition that must be satisfied by orientifold models.

1 A Puzzle

Orientifolds are a generalization of orbifolds, allowing the construction of interesting string theories based on free world-sheet fields. Although discovered some time ago [3], they have recently attracted renewed interest because they are related by string dualities to many other vacua. In particular, there has been a series of papers constructing models of this type in $d = 6$ with $N = 1$ supersymmetry [4, 5, 1, 2].[1] In this note we would like to resolve a small puzzle arising from some of this work. This will lead us to a new orientifold construction and also to a world-sheet consistency condition not previously noticed, though fortunately satisfied by all the models of refs. [4, 5, 1, 2]. It also provides a nice illustration of the recent idea of using D-branes as probes of spacetime geometry [8].

The puzzle is this. Refs. [4, 1, 2] all construct what appears to be the Type I string on a $K3$ orbifold, in that one twists the IIB theory on T^4 by world-sheet parity Ω (producing the Type I string) and by a spacetime $\mathbf{Z}_N$ rotation (producing K3). For the $\mathbf{Z}_2$ case [4], the spectrum agrees with that of the Type I string on a smooth $K3$.[2] However, for $N \geq 3$ the orientifold spectrum does not agree with that on smooth $K3$. The latter, like its heterotic dual, always has a single antisymmetric tensor multiplet, while in the orientifold an extra m tensors live at each $\mathbf{Z}_{2m+1}$ or $\mathbf{Z}_{2m+2}$ fixed point [1, 2].

The cause of the discrepancy is that the parity operator Ω of the orientifold theory is not given by the limit of the Ω of the smooth theory.[3] Let us consider first the Type IIB theory at a $\mathbf{Z}_N$ orbifold point (also known as

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[1]Orientifolds with $d = 6$ $N = 1$ supersymmetry were first found in refs. [6]. It is likely that some of the models found in these papers are identical to models in refs. [4, 5, 1, 2]. However the puzzle we consider depends on the spacetime picture, and so is not evident in these fermionic constructions. For further background on D-branes and orientifold see ref. [7].

[2]The blowing up of the orbifold singularities was discussed in detail in ref. [9].

[3]This is also under study by E. Gimon and C. Johnson [10], by A. Dabholkar and J. Park [11], and by J. Blum.

an A_{N-1} singularity). The closed string Hilbert space has twisted sectors labeled by $k = 1, \ldots N - 1$,

$$Z^{3,4}(2\pi) = \alpha^k Z^{3,4}(0), \qquad \alpha = e^{2\pi i/N}, \qquad (1.1)$$

where $Z^l = X^{2l} + iX^{2l+1}$. The right-moving NS sector contains two states which are singlets under the massless $SO(4)$ little group, while the right-moving R sector contains a doublet. The left-moving sectors are the same, so the full spectrum is

$$(2 \cdot \mathbf{1} + \mathbf{2}) \times (2 \cdot \mathbf{1} + \mathbf{2}) = 5 \cdot \mathbf{1} + \mathbf{3}. \qquad (1.2)$$

These five singlets plus anti-self-dual tensor form a tensor multiplet of Type IIB supergravity, call it $\mathcal{T}_k$, which decomposes to a tensor multiplet and hypermultiplet of Type I. Thus there are $N - 1$ tensor multiplets associated with the fixed point. This is in agreement with the limit of the smooth $K3$ spectrum. At the fixed point there are $N - 1$ collapsed two-spheres. Each gives rise to a an anti-self-dual tensor (from the self-dual four-form of the IIB theory), plus three moduli from the metric and two theta parameters, from the R and NS two-forms.

Under the IIB parity operator Ω, the metric and R two-form are even and the four-form and the NS two-form are odd. Projection onto $\Omega = +1$ thus leaves $N - 1$ hypermultiplets at the fixed point. The parity operator of the orientifold theory, call it Ω', acts differently [1, 2]. Reversing the orientation of the string changes the α^k twist (1.1) to $\alpha^{-k} = \alpha^{N-k}$. Except when $k = N - k$, this is off-diagonal and so one of the two linear combinations $\mathcal{T}_k \pm \mathcal{T}_{N-k}$ is even and the other odd. For $k = N - k$ (so that N is even), Ω' takes $\mathcal{T}_{N/2}$ to itself, and the tensor and one NS scalar are odd and the other four scalars even. Forming the orientifold by projecting onto $\Omega' = +1$ then leaves $m - 1$ tensor multiplets and m hypermultiplets for $\mathbf{Z}_{2m}$ and m tensor multiplets and m hypermultiplets for $\mathbf{Z}_{2m+1}$. Only for $\mathbf{Z}_2$ does $\Omega = \Omega'$.

Evidently $\Omega' = \Omega J$ where J is some symmetry of the orbifold CFT which only acts on the fields near the fixed point. In the rest of the paper we will

study this J in more detail, not just in the $\mathbf{Z}_N$ orbifold limit but for the full moduli space of Ω'-invariant $K3$'s. To make the discussion simple we focus on a $\mathbf{Z}_3$ fixed point. From the above discussion, this has two $\Omega = +1$ hypermultiplets, only one of which has $\Omega' = +1$. Consider the ALE space, the local region of $K3$, obtained by turning on the $\Omega' = +1$ moduli. This cannot be smooth as there would be no candidate for J, a symmetry which must act trivially at long distance from the blown up fixed point. But there is indeed a family of singular ALE spaces which includes the $\mathbf{Z}_3$ orbifold and which is parameterized by one hypermultiplet. These spaces have A_1 singularities, $\mathbf{Z}_2$ orbifold points.

From the orbifold point of view it is hard to see a $\mathbf{Z}_3$ fixed point deform into a $\mathbf{Z}_2$ fixed point, but from other points of view it is simple. The metric for a general $\mathbf{Z}_N$ ALE space is of the form [12]

$$ds^2 = V^{-1}(dt - \mathbf{A} \cdot d\mathbf{y})^2 + V d\mathbf{y} \cdot d\mathbf{y}$$
$$V = \sum_{i=0}^{N-1} \frac{1}{|\mathbf{y} - \mathbf{y}_i|}, \qquad \boldsymbol{\nabla} V = \boldsymbol{\nabla} \times \mathbf{A} \tag{1.3}$$

where $\mathbf{y}$ is a three-vector and t has period 4π. The $3(N-1)$ moduli mentioned earlier are just the $N - 1$ differences $\mathbf{y}_i - \mathbf{y}_0$; by a translation one can set $\mathbf{y}_0 = 1$. For $N = 3$, the $\mathbf{Z}_3$ orbifold singularity occurs when the three 'charges' are coincident, $\mathbf{y}_0 = \mathbf{y}_1 = \mathbf{y}_2$. Pulling one charge away and leaving two coincident leaves a $\mathbf{Z}_2$ fixed point, and these singular ALE spaces are indeed parameterized by one hypermultiplet.

There are no further hypermultiplets to associate with blowing up the $\mathbf{Z}_2$ fixed point but there is a tensor multiplet, so this is different from the $\mathbf{Z}_2$ fixed points of ref. [4]. The difference is now clear: the Ω' projection is just keeping the opposite states from Ω at the fixed point, so that J is the $\mathbf{Z}_2$ symmetry which is -1 in the $\mathbf{Z}_2$-twisted sector and $+1$ in the untwisted sector.[4]

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[4]This is after the $\mathbf{Z}_3$ singularity is partly resolved into a $\mathbf{Z}_2$ singularity. On the original

In the next section we study Ω' orientifolds of free field theory. In the final section we verify the above argument about the ALE geometry by using D-branes to measure the blown-up metric directly. By the same method we find that the generic blowup of the $\mathbf{Z}_{2m+1}$ and $\mathbf{Z}_{2m+2}$ singularities of refs. [1, 2] leaves m separate A_1 singularities, each with an associated tensor multiplet.

2 A New $\mathbf{Z}_2$ Orientifold

Now let us study the new $\mathbf{Z}_2$ fixed point in isolation. Start with the IIB string in 10 dimensions. Twist by a reflection R of $X^{6,7,8,9}$ to produce an orbifold point at the origin, and then twist by ΩJ. Note that this is akin to an asymmetric orbifold, in that the symmetry J does not exist until after the first twist. Note also that if the second twist is by J rather than ΩJ it simply undoes the R-twist. Projecting onto $J = +1$ removes the R-twisted states from the spectrum, while the J-twisted states are by definition of J those which have a branch cut relative to the R-twisted states—that is, they are the R-odd states that were removed by the first projection.

In order for the ΩJ twist to be consistent, ΩJ must be conserved. For purely closed string processes this follows because Ω and J are both conserved. It is also necessary to check the closed-to-open transition: ΩJ as defined in the closed string and open string sectors must be conserved by this transition. This must be true for all orbifold and orientifold twists in open string theory, and is the missing consistency condition mentioned earlier. We will therefore also reexamine the earlier models.

To be specific we consider the transition between the closed string RR ground state and the open string vector. The analytic parts of the relevant

orbifold with $\mathbf{Z}_3$ singularity, J must interchange the sectors twisted by α and α^2 while leaving the untwisted sectors invariant. This symmetry is not manifest, for example it is not a symmetry of operator products involving $\mathbf{Z}_3$ non-invariant operators, but must be present in the orbifold CFT.

vertex operators are

$$
\begin{aligned}
V_\alpha &= e^{-\phi/2} S_\alpha \\
T'_\alpha &= e^{-\phi/2} S'_\alpha T \\
V^\mu &= e^{-\phi} \psi^\mu
\end{aligned}
\tag{2.1}
$$

for the untwisted R ground state, twisted R ground state, and untwisted NS vector respectively. Here ϕ is the bosonized ghost [13], S_α the spin field (spinor indices are ten-dimensional on unprimed objects and six-dimensional on primed), and T the internal part of the twisted R ground state. All of these operators are weight $(1,0)$, with the corresponding $(0,1)$ operators denoted by a tilde. The relevant closed-to-open amplitudes on the unit disk are determined purely by Lorentz invariance,

$$
\begin{aligned}
\langle V_\alpha(0) \tilde{V}_\beta(0) V^\mu(1) \rangle &= \Gamma^\mu_{\alpha\beta} \\
\langle T'_\alpha(0) \tilde{T}'_\beta(0) V^\mu(1) \rangle &= \Gamma'^\mu_{\alpha\beta}.
\end{aligned}
\tag{2.2}
$$

Now consider the effect of orientation reversal Ω. This takes

$$
\begin{aligned}
V_\alpha \tilde{V}_\beta &\rightarrow \tilde{V}_\alpha V_\beta = -V_\beta \tilde{V}_\alpha \\
T'_\alpha \tilde{T}'_\beta &\rightarrow \tilde{T}'_\alpha T'_\beta = -T'_\beta \tilde{T}'_\alpha \\
V^\mu &\rightarrow -V^\mu.
\end{aligned}
\tag{2.3}
$$

Note that the R vertex operators anticommute, being spacetime spinors. That the photon V^μ is Ω-odd is familiar, though less obvious in the -1 picture (2.1) than in the 0 picture where it is a tangent derivative. The ten-dimensional Γ matrices are symmetric so the untwisted amplitude (2.2) is even under Ω, but the six-dimensional Γ matrices are antisymmetric and the twisted amplitude is odd. The full amplitude also contains a Chan-Paton factor. The twist R acts on the Chan-Paton factors as a matrix γ_R, so the Chan-Paton trace for the twisted amplitude contains a factor γ_R: it is of the form

$$
\mathrm{Tr}(\gamma_R \lambda_1 \dots \lambda_n).
\tag{2.4}
$$

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Parity Ω takes $\lambda_i \to (\gamma_\Omega^{-1} \lambda_i \gamma_\Omega)^T$ and reverses the order $1, \ldots, n$. This is equivalent to $\gamma_R \to \gamma_\Omega \gamma_R^T \gamma_\Omega^{-1}$. Conservation of Ω in the full amplitude then requires

$$\gamma_R = -\gamma_\Omega \gamma_R^T \gamma_\Omega^{-1}. \tag{2.5}$$

This was not imposed explicitly in ref. [4], though the condition that $[\Omega, R] = 0$ hold on the Chan-Paton factors does imply eq. (2.5) up to a sign. In fact, the sign is correct for the model of ref. [4]. It is also correct for the $\mathbf{Z}_2$ twist fields of refs. [1, 2], because this sector is the same as in ref. [4] for all models. For twists other than $\mathbf{Z}_2$ the new condition is not as interesting. For these, Ω takes the twist g into a different twist g^{-1} so it governs which linear combination of the two sectors appears. This is necessary to get the correct vertex operators, but does not affect the spectrum. Similarly in ref. [5], Ω is replaced by an operator ΩS which acts off-diagonally on the R-fixed points.

Now consider the operator ΩJ. By definition, this has an extra minus sign in its action on $T'_\alpha \tilde{T}'_\beta$, so the above argument leads to

$$\gamma_R = +\gamma_{\Omega J} \gamma_R^T \gamma_{\Omega J}^{-1}. \tag{2.6}$$

In the nine-brane sector, cancellation of the ten-form tadpole requires as always 32 nine-brane indices with $\gamma_{\Omega J,9} = 1$ in an appropriate basis. Then $\gamma_{R',9}$ is now symmetric, and so is $\gamma_{\Omega J R,9} \propto \gamma_{R,9} \gamma_{\Omega J,9}$. The Chan-Paton algebra then implies that $\gamma_{\Omega J R,5}$ is antisymmetric, the opposite of $\gamma_{\Omega R,5}$ in ref. [4]. Examining the tadpoles of ref. [4], this changes the sign of the cross-term in the untwisted six-form tadpole so that the fixed point has the opposite charge from the usual $\mathbf{Z}_2$ fixed point: $+\frac{1}{2}$ times the five-brane charge for the new fixed point (call it type B) versus $-\frac{1}{2}$ for the old (call it type A).

The vanishing of the twisted tadpole is not automatic as it is at the usual fixed point. At the latter the Ω projection removes the dangerous twisted sector six-form along with the two-form, but the ΩJ projection leaves both. So if for example there are no five-branes at some fixed point, the tadpole condition of ref. [4] implies that $\text{Tr}(\gamma_{R,9}) = 0$ for the ΩJ projection. The

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symmetry of $\gamma_{R,9}$ and the surviving orthogonal change of basis after setting $\gamma_{\Omega J,9} = 1$ then allow to take

$$\gamma_{R,9} = \begin{bmatrix} I_{16} & 0 \\ 0 & -I_{16} \end{bmatrix}. \tag{2.7}$$

We have studied the fixed point in a non-compact space but now let us build a compact model. The six-form charges found above do not allow all sixteen fixed points to be of the new type as all six-form sources would have the same sign, but they suggest a model with eight fixed points of type A, eight of type B, and no five-branes. Indeed this is possible. Take $K3$ to be a hypercube of side 2, so the coordinates the fixed point are all 0 or 1. Consider the eight fixed points with $X^6 = 0$. The product J_0 of the eight separate J's is not conserved. However, the transition of a string from a fixed point with $X^6 = 0$ to one with $X^6 = 1$ produces also a string with winding number w_6 odd (in the orbifold w_6 is only defined mod 2), so that $J_0(-1)^{w_6}$ is conserved. To see this, define R as the reflection which leaves $(0,0,0,0)$ invariant. Then $(1,0,0,0)$ is left invariant by $T_6 R$ where T_6 is a translation by 2 units in the X^6 direction. A transition from a state of monodromy R to one of monodromy $T_6 R$ produces also a string of monodromy $T_6^{-1} \cong T_6$, i. e., odd winding number.

Note that it is not possible in general to mix fixed points of different types in an arbitrary way. It is only consistent to project (gauge) on an operation which is a symmetry of string theory. One sees from the above that fixed points of types A nd B can only be combined in groups of eight.

The prescription then is to project the $K3$ orbifold of the IIB theory by $\Omega J = \Omega J_0(-1)^{w_6}$ and add in the usual 32 nine-brane indices. The symmetries act on the Chan-Paton matrices as $\gamma_{\Omega J} = 1$ and γ_R given by eq. (2.7), while the condition that the eight fixed points with $X^6 = 1$ be of type A require that $\gamma_{T_6 R}$ be as in ref. [4],

$$\gamma_{T_6 R} = \begin{bmatrix} 0 & I_{16} \\ -I_{16} & 0 \end{bmatrix}. \tag{2.8}$$

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This implies a nontrivial Wilson line $\gamma_{T_6} = \gamma_{T_6 R} \gamma_R^{-1}$. The resulting model satisfies all algebraic and tadpole conditions.

The untwisted closed string sector contributes the usual gravitational multiplet, tensor multiplet, and four hypermultiplets. The twisted sectors obviously contribute eight tensor multiplets and eight hypermultiplets. The open string gauge group is broken from $SO(32)$ to $SO(16) \times SO(16)$ by γ_R and then to $SO(16)$ by γ_{T_6}. The open string spectrum also includes a hypermultiplet in the adjoint of $SO(16)$.

This is the same spectrum found in the models of ref. [5]. Those models had five-branes but no nine-branes. Not surprisingly then, the model found here is equivalent to those under T-duality on the $X^{6,7,8,9}$ axes. The T-duality acts on the fixed point states as a discrete Fourier transform, taking a fixed point with coordinates X^m into a linear combination of all fixed points, the fixed point with coordinates Y^m being weighted by $2^{-4/2}(-1)^{X \cdot Y}$. It follows that $J_0 = -(-1)^{X^6}$ maps to the translation $Y^6 \to Y^6 + 1$ of the fixed point states, up to a sign that can be absorbed in the definition of the states. The factor $(-1)^{w_6}$ maps to $(-1)^{n_6}$, which translates untwisted states by half the lattice spacing. Thus $J_0(-1)^{w_6}$ maps to the full operator $T_6^{1/2}$ for translation by half the lattice spacing. In the notation of ref. [5], $T_6^{1/2} = RS$ and the orientifold groups $\{1, R, \Omega S, \Omega RS\}$ of the two models are the same. Since R here maps to R of ref. [5], our model is dual to symmetric solution of that paper. But if we translate our model $X^6 \to X^6 + 1$ and then take the T-dual, $T_6 R$ of our model would map to R of that paper, giving the antisymmetric solution. It must be that the two solutions of ref. [5] are equivalent under a redefinition by the image of $T_6^{1/2}$, namely $J_0(-1)^{w_6}$. One finds that this in indeed the case.[5]

The models of ref. [5] also have two kinds of fixed point, neither of which is A or B. Half the fixed points are ordinary orbifold points, fixed by R but

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[5]In verifying this, note the in models with five-branes only, open strings should be regarded as having winding number $w_6 = (X^6(\pi) - X^6(0))/2\pi R_6$.

no operation involving Ω. These fixed points have no six-form charge, there being no associated crosscap, and as discussed in the introduction have a tensor and a hypermultiplet. The other type are fixed only by an operation ΩS, equivalent to ΩR. These must have charge -1, by overall neutrality of the model, and have no associated twisted states. Finally, Dabholkar and Park have recently found yet another kind of $\mathbf{Z}_2$ singularity in orientifold models, having a tensor multiplet but six-form charge $-\frac{1}{2}$. This is based on the projection ΩJ but with a different action on the Chan-Paton factors. In particular, the operator Ω^2 is $+1$ in the 59 open string sector, having the minus sign noted in ref. [4] plus an additional minus sign because the 59 sector has half-integer rather than integer masses.

3   ALE Geometry

We can directly test our argument about the geometry of the Ω'-invariant $K3$'s by using the D-string as a probe, as recently proposed by Douglas [8]. All results in the present section are already implicit in refs. [14, 15, 16], but the D-probe idea seems very promising and so it is worthwhile to work out this simple example explicitly.

We start with the $\mathbf{Z}_2$ ALE space. Consider the IIB theory with a $\mathbf{Z}_2$ orbifold point (or more precisely six-plane) at $X^{6,7,8,9} = 0$, and add a D-string in this plane at $X^{2,\dots,9} = 0$. In order for the string to be able to move off the fixed plane it needs two Chan-Paton indices, for the string and its $\mathbf{Z}_2$ image. Since R takes the D-string into its image, γ_R is the Pauli matrix σ^1. The massless NS spectrum of the string is then (in terms of the 0 picture vertex operators)

$$
\begin{aligned}
&\partial_t X^\mu \sigma^{0,1}, && \mu = 0, 1 \\
&\partial_n X^i \sigma^{0,1}, && i = 2, 3, 4, 5 \\
&\partial_n X^m \sigma^{2,3}, && m = 6, 7, 8, 9.
\end{aligned}
\tag{3.1}
$$

These are respectively a gauge field, the position of the string within the six-plane, and the transverse position. Call the corresponding D-string fields A^μ, x^i, x^m, all 2×2 matrices. The bosonic action is the $d = 10$ $U(2)$ Yang-Mills action, dimensionally reduced and R-projected (which breaks the gauge symmetry to $U(1) \times U(1)$). In particular, the potential is

$$U = 2 \sum_{i,m} \mathrm{Tr}([x^i, x^m]^2) + \sum_{m,n} \mathrm{Tr}([x^m, x^n]^2). \tag{3.2}$$

The moduli space thus has two branches. On one, $x^m = 0$ and $x^i = u^i \sigma^0 + v^i \sigma^1$. This corresponds to two D-strings moving independently in the plane, with positions $u^i \pm v^i$. The gauge symmetry is unbroken, giving independent $U(1)$'s on each D-string.[6] On the other branch, x^m is nonzero and $x^i = u^i \sigma^0$. The σ^1 gauge invariance is broken and so by gauge choice $x^m = w^m \sigma^3$. This corresponds to the D-string moving off the fixed plane, the string and its image being at $(u^i, \pm w^m)$.

Now let us turn on twisted-sector moduli. Define complex q^m by $x^m = \sigma^3 \mathrm{Re}(q^m) + \sigma^2 \mathrm{Im}(q^m)$, and define two doublets,

$$\Phi_0 = \begin{pmatrix} q^6 + iq^7 \\ q^8 + iq^9 \end{pmatrix}, \qquad \Phi_1 = \begin{pmatrix} \bar{q}^6 + i\bar{q}^7 \\ \bar{q}^8 + i\bar{q}^9 \end{pmatrix}. \tag{3.3}$$

These have charges ± 1 respectively under the σ^1 $U(1)$. The three NSNS moduli can be written as a vector $\mathbf{D}$, and the potential is proportional to

$$(\Phi_0^\dagger \boldsymbol{\tau} \Phi_0 - \Phi_1^\dagger \boldsymbol{\tau} \Phi_1 + \mathbf{D})^2, \tag{3.4}$$

where the Pauli matrices are now denoted τ^a to emphasize that they act in a different space. This reduces to the second term of the earlier potential (3.2) when $\mathbf{D} = 0$. Its form is determined by supersymmetry, and the trilinear

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[6]These D-strings can actually be regarded as collapsed three-branes wrapped on the orbifold point. They couple to the corresponding RR field, the twisted-sector tensor. Because the θ-parameter from the NS sector is nonzero [17], they also carry the untwisted six-form charge. When the theta parameter is tuned to zero these strings become tensionless [15].

coupling between the twisted sector field and two open string fields was demonstrated in the appendix to ref. [16].

For $\mathbf{D} \neq 0$ the orbifold point is blown up. The moduli space of the D-string is simply the set of possible locations, that is, the blown up ALE space.[7] The z^m contain eight scalar fields. Three are removed by the $\mathbf{D}$-flatness condition, that the potential vanish, and a fourth is a gauge degree of freedom, leaving the expected four moduli. In terms of supermultiplets, the system has the equivalent of $d = 6$ $N = 1$ supersymmetry. The D-string has two hypermultiplets and two vector multiplets, which are Higgsed down to one hypermultiplet and one vector multiplet.

The idea of ref. [8] is that the metric on this moduli space, as seen in the kinetic term for the D-string fields, should be the smoothed ALE metric. It is straightforward to verify this. Define

$$\mathbf{y} = \Phi_0^\dagger \boldsymbol{\tau} \Phi_0. \tag{3.5}$$

This gives three coordinates on moduli space. The fourth coordinate t can be defined

$$t = 2 \arg(\Phi_{0,1} \Phi_{1,1}). \tag{3.6}$$

The period of t is 4π because of the orbifold projection. The $\mathbf{D}$-flatness condition implies that

$$\Phi_1^\dagger \boldsymbol{\tau} \Phi_1 = \mathbf{y} + \mathbf{D}, \tag{3.7}$$

and Φ_0 and Φ_1 are determined in terms of $\mathbf{y}$ and t, up to gauge choice.

The original metric is $d\Phi_0^\dagger d\Phi_0 + d\Phi_1^\dagger d\Phi_1$, but we need to project this into the space orthogonal to the $U(1)$ gauge transformation.[8] The result is

$$ds^2 = d\Phi_0^\dagger d\Phi_0 + d\Phi_1^\dagger d\Phi_1 - \frac{(\omega_0 + \omega_1)^2}{4(\Phi_0^\dagger \Phi_0 + \Phi_1^\dagger \Phi_1)} \tag{3.8}$$

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[7]Note that the branch of the moduli space with $v^i \neq 0$ is no longer present.

[8]This whole construction, imposing the $\mathbf{D}$-flatness conditions and making the gauge identification, is known as the hyper-Kähler quotient [14].

where

$$\omega_i = i(\Phi_i^\dagger d\Phi_i - d\Phi_i^\dagger \Phi_i) \tag{3.9}$$

.

It is now straightforward to express the metric in terms of $\mathbf{y}$ and t using the identity $(\alpha^\dagger \tau^a \beta)(\gamma^\dagger \tau^a \delta) = 2(\alpha^\dagger \delta)(\gamma^\dagger \beta) - (\alpha^\dagger \beta)(\gamma^\dagger \delta)$ for arbitrary doublets $\alpha, \beta, \gamma, \delta$. This implies, for example,

$$\Phi_0^\dagger \Phi_0 = |\mathbf{y}|, \qquad \Phi_1^\dagger \Phi_1 = |\mathbf{y} + \mathbf{D}|,$$
$$d\mathbf{y} \cdot d\mathbf{y} = |\mathbf{y}|d\Phi_0^\dagger d\Phi_0 - \omega_0^2 = |\mathbf{y} + \mathbf{D}|d\Phi_1^\dagger d\Phi_1 - \omega_1^2, \tag{3.10}$$

and the metric is readily found to be of the ALE form (1.3) with $\mathbf{y}_0 = 0$, $\mathbf{y}_1 = \mathbf{D}$, up to a normalization that can be absorbed in a coordinate transformation. In particular, the vector potential is

$$\mathbf{A}(\mathbf{y}) \cdot d\mathbf{y} = |\mathbf{y}|^{-1}\omega_0 + |\mathbf{y} + \mathbf{D}|^{-1}\omega_1 + dt, \tag{3.11}$$

and the field strength is readily obtained by taking the exterior derivative and using the identity $\epsilon^{abc}(\alpha^\dagger \tau^b \beta)(\gamma^\dagger \tau^c \delta) = i(\alpha^\dagger \tau^a \delta)(\gamma^\dagger \beta) - i(\alpha^\dagger \delta)(\gamma^\dagger \tau^a \beta)$.

This all extends to the $\mathbf{Z}_N$ case. In order to move away from the fixed point the D-string needs N Chan-Paton indices, with the Z_N matrix $\gamma_\alpha = a$ taking each index into the next. Define another $N \times N$ matrix b with the properties $ab = \alpha ba$ ($\alpha = e^{2\pi i/N}$), $b^N = 1$, which together with a^N define a and b up to change of basis. The open string states in a convenient basis are

$$\begin{aligned}
\partial_t X^\mu \lambda, & \quad a^{-1}\lambda a = \lambda & \Rightarrow & \quad \lambda = P_r \\
\partial_n X^i \lambda, & \quad a^{-1}\lambda a = \lambda & \Rightarrow & \quad \lambda = P_r \\
\partial_n Z^l \lambda, & \quad a^{-1}\lambda a = \alpha^{-1}\lambda & \Rightarrow & \quad \lambda = bP_r \\
\partial_n \bar{Z}^l \lambda, & \quad a^{-1}\lambda a = \alpha \lambda & \Rightarrow & \quad \lambda = b^{-1}P_r \tag{3.12}
\end{aligned}$$

where

$$P_r = N^{-1/2} \sum_{k=0}^{N-1} \alpha^{rk} a^k \tag{3.13}$$

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are projection operators. Call the corresponding fields A_r^μ, x_r^μ, Φ_r^l, $\bar{\Phi}_r^l$ with $r = 0, \ldots, N-1$. With the lower index suppressed these are $N \times N$ matrices. The covariant derivative is

$$D_\mu \Phi = \partial_\mu \Phi + i[A_\mu, \Phi]. \tag{3.14}$$

Noting that $P_r b = b P_{r+1}$, this implies

$$D_\mu \Phi_r = \partial_\mu \Phi_r + i(A_{r-1,\mu} - A_{r,\mu})\Phi_r, \tag{3.15}$$

with $r = N \equiv r = 0$. The N $U(1)$ **D**-flatness conditions require

$$\Phi_{r+1}^\dagger \boldsymbol{\tau} \Phi_{r+1} - \Phi_r^\dagger \boldsymbol{\tau} \Phi_r = \mathbf{D}_r. \tag{3.16}$$

Note that this implies that there are $N-1$ **D**-terms, as the sum must vanish. The metric is then readily written in ALE form, with

$$\mathbf{y}_i = \sum_{r=0}^{i-1} \mathbf{D}_r. \tag{3.17}$$

Now we can return to our original purpose, which was to identify the Ω'-invariant ALE spaces. Recall above that the matrix a is interpreted as connecting each D-string with its image rotated by α. A Chan-Paton factor proportional to a^k is then an open string with one end at one image and the other rotated by α^k. Orientation reversal, whether Ω or Ω', switches the endpoints and so takes this into a string with Chan-Paton factors a^{-k}. In terms of the projection operators this is $P_r \to P_{N-r}$, and so

$$\Omega' : \quad \Phi_r \to \Phi_{N-r}, \quad \mathbf{D}_r \to -\mathbf{D}_{N-r-1}. \tag{3.18}$$

It follows that for $N = 3$, requiring the twisted background to be Ω'-even implies that $\mathbf{D}_1 = 0$, $\mathbf{D}_2 = -\mathbf{D}_0$, and so $\mathbf{y}_1 = \mathbf{y}_2 \neq \mathbf{y}_0$ as conjectured. Similarly for the general Z_N fixed point, one finds that $\mathbf{y}_i = \mathbf{y}_{N-i}$, leaving m collapsed two-spheres for $N = 2m + 1$ or $2m + 2$.

It would be interesting to understand the strong-coupling behavior of the theory near the ALE singularity. Away from the singularity it is strongly coupled Type I and so weakly coupled heterotic $SO(32)$, but there is no perturbative background of the heterotic string with extra tensors. In this connection we should note that there have been many recent discussions of extra tensors in the contexts of M-theory and F-theory; it is not clear whether these are directly relevant, since the *local physics* near but not at the singularity is just the heterotic string.

Acknowledgments

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